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A unified approach to acoustic fields induced in inhomogeneous spheres by external sources

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Abstract

The evaluation of acoustic fields induced in inhomogeneous spheres by external sources, based on volume integral equations, is simplified considerably using Ivakin's integral equation and the use of the well-known expansion for the Green dyadic. The method presented here overcomes the difficulties of calculation arising from the possible step discontinuities in the density, allowing a quick and accurate evaluation of the fields. So the simple hybrid (analytical–numerical) scalar method, developed previously for spheres with variable, even discontinuous compressibility, is easily extended to the general case of inhomogeneous density in a manner requiring the least possible analytical and computational effort.

1. Introduction

The evaluation of acoustic fields induced in inhomogeneous bodies by external sources [1–6] is a basic problem whose solution finds practical applications to questions related to radiation hazards, to the setting of reliable safety field strength limits, geophysical explorations, seismic engineering and underwater acoustics. For general shape inhomogeneities, the analysis is performed using numerical schemes [3–6]. However, in the case of inhomogeneous spheres analytical procedures are, up to a certain point, possible. So the study of inhomogeneous spheres is, apart from its own interest, useful in the assessment of the quality and efficiency of the previously mentioned numerical methods.

The problem of interest is as follows: An incident acoustic pressure field $\Phi^{\mathrm{inc}}(\mathbf{r})$ impinges on an inhomogeneous spherical volume V of radius a and induces the total (unknown) scalar pressure field $\Phi(\mathbf{r})$ (see Fig. 1). A harmonic time dependence $\exp(\mathrm{i}\omega t)$ is assumed. The differential equation satisfied by $\Phi(\mathbf{r})$ is [3–6]

$$\nabla \cdot \left[\frac{\nabla \Phi(\mathbf{r})}{\rho(\mathbf{r})}\right] = -\omega^2 b(\mathbf{r})\Phi(\mathbf{r}), \tag{1}$$

where $\rho(\mathbf{r})$, $b(\mathbf{r})$ are the varying density and compressibility of the medium in V; V is surrounded by a medium of constant ρ_0, b_0 values. We note in passing that this equation appears also in Ref. [2], but with a typographical error on the right-hand side. The boundary conditions are continuity at all interfaces for both the field $\Phi(\mathbf{r})$ and for the normal component of particle velocity, $\nabla\Phi(\mathbf{r})/\rho(\mathbf{r})$.

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Nomenclature

a radius of the inhomogeneous sphere
$b(\mathbf{r})$ compressibility of the inhomogeneous medium
b_0 compressibility of the free space
$g(\mathbf{r},\mathbf{r}')$ scalar Green function, see Eq. (3)
$\mathbf{G}(\mathbf{r},\mathbf{r}')$ dyadic Green function, see Eq. (9)
$h_n^{(2)}(x)$ spherical Hankel function of the second kind $j_n - \mathrm{i}n_n$ of order n
$I(\mathbf{X},\mathbf{Y})$ overlap integral of $\mathbf{X}$ and $\mathbf{Y}$ over the volume of the sphere, see for example Eqs. (A.1), (A.4), and (A.9)
$\mathbf{I}$ identity dyadic
$j_n(x)$ spherical Bessel function of order n
k_0 the wavenumber of the free space $\omega(b_0\rho_0)^{1/2}$
$\mathbf{L}^S_{mn\ell}$ the non-orthogonal $(a/\gamma_{mn\ell})\nabla\Psi_{mn\ell}$ vectors obtained through $\nabla\Phi$ in Eq. (21)
$\mathbf{M}^{(i)}_{mn}, \mathbf{N}^{(i)}_{mn}, \mathbf{L}^{(i)}_{mn}$ spherical eigenvectors of the ith kind defined in Eqs. (11)–(13) for continous k. The upperscripts $i=1$ or 4 imply the use of j_n or $h_n^{(2)} = j_n - \mathrm{i}n_n$, respectively, for the radial dependence. When $i=1$ the upperscipt is usually omitted.
$\hat{\mathbf{M}}_{mn}, \hat{\mathbf{N}}_{mn}, \hat{\mathbf{L}}_{mn}$ same as $\mathbf{M}_{mn}$, etc., where the angular part of $\mathbf{M}_{mn}$, etc. has been replaced by its complex conjugate
$\mathbf{M}_{mn\ell}, \mathbf{N}_{mn\ell}, \mathbf{L}_{mn\ell}$ spherical eigenvectors for the spherical cavity, with discrete values of $k = \gamma^M_{mn\ell}/\alpha,\ \gamma^N_{mn\ell}/\alpha,\ \gamma^L_{mn\ell}/\alpha,\ \ell = 1,2,\ldots,$ respectively
$n_n(x)$ spherical Neumann function of order n
$\mathbf{P}_{mn}, \mathbf{B}_{mn}, \mathbf{C}_{mn}$ surface spherical harmonic vectors defined in Eqs. (14)–(16)
$\mathbf{S}$ matrix of the expansion coefficients of $\mathbf{L}^S_{mn\ell}$ in terms of $\mathbf{L}_{mn\ell}$ see Eq. (42)
$t_{nm}, t^M_{nm}, t^N_{nm}, t^L_{nm}$ constants arbitrarily chosen for scalar expansions (19) and (20) see Eq. (22) and the vector ones in Eq. (21), see Eqs. (A.2), (A.7), and (A.10)
$T_{mn\ell}(k_0)$ expansion coefficients of $j_n(k_0 r)$ see Eq. (29)
A, B, Γ, Δ, Z expansion coefficients see Eqs. (19)–(21)
$\gamma_{mn\ell}, \gamma^M_{mn\ell}, \gamma^N_{mn\ell}, \gamma^L_{mn\ell}$ roots of the eigenvalue problems defined in Eqs. (22), (A.2), (A.7), and (A.10)
$\rho(\mathbf{r})$ density of the inhomogeneous medium
ρ_0 density of the free space
$\Phi(\mathbf{r}), \Phi^{\mathrm{inc}}(\mathbf{r})$ total and incident acoustic pressure field, respectively
$\Psi_{mn}(k,\mathbf{r})$ solution of the scalar Helmholtz equation $(\nabla^2 + k^2)\Psi = 0$, see Eq. (11)
$Y_{mn}(\theta,\phi)$ surface spherical harmonics $P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$

In two previous papers [1,2], the treatment was based on a volume integral equation derived by a technique proposed by Chew [3]. However, this equation, like that used by Martin [7], contained the derivative of the density in its integrand. To carry out the calculations correctly we had split in Ref. [2] the non-continuous part

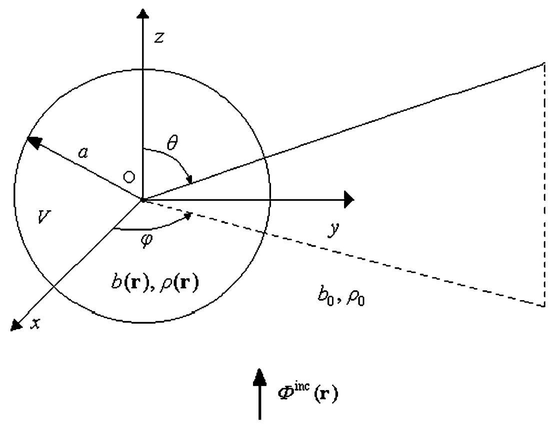

Fig. 1. Geometry of the problem.

of the density with the use of step functions and this led (through differentiation) to Dirac δ-functions. The necessity for a special treatment of step discontinuities in the density is, also, the basis of Martin's approach [7], who adds a term proportional to the density discontinuity in the integral equation.

In this paper, we begin our investigation using another integral equation which has been derived by Ivakin [4, Eq. (5)]:

$$\Phi(\mathbf{r}) = \Phi^{\mathrm{inc}}(\mathbf{r}) + \int_V k_0^2 \left[\frac{b(\mathbf{r}')}{b_0} - 1\right] \Phi(\mathbf{r}') g(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' - \int_V \left[\frac{\rho_0}{\rho(\mathbf{r}')} - 1\right] \nabla'\Phi(\mathbf{r}') \cdot \nabla' g(\mathbf{r},\mathbf{r}')\,\mathrm{d}V'. \tag{2}$$

No restrictions have been imposed on $\rho(\mathbf{r})$. Clearly, this equation overcomes the previously mentioned difficulty, since it does not contain any derivative of the density. A simple check for the validity of Eq. (2) can be done in the case of a sphere of constant density $\rho_1 \neq \rho_0$. This will be done later. In Eq. (2), $g(\mathbf{r},\mathbf{r}')$ is the scalar Green's function of free space

$$g(\mathbf{r},\mathbf{r}') = \mathrm{e}^{-\mathrm{i}k_0 R}/4\pi R, \quad R = |\mathbf{r}-\mathbf{r}'|, \quad k_0 = \omega(b_0\rho_0)^{1/2}, \tag{3}$$

which obeys the following equation:

$$(\nabla^2 + k_0^2) g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{4}$$

In what follows we will apply the convenient method developed in Refs. [1,2] to solve integral equation (2). The essence of the approach consists in expanding the radial part of Φ in a Dini series. We have explained in Refs. [1,2], that the Dini expansions in terms of Bessel functions have a clear advantage over other possible orthogonal sets of functions, since, among other things, they converge faster and allow an analytical evaluation of certain integrals involving the Green's function (of free space here) [8,9]. However, the approach it is not devoid of difficulties, which arise from the appearance of the term $\nabla\Phi \cdot \nabla g$ in the integrand of Eq. (2). The steps required to transform the integral equation into a more convenient form are presented in Section 2. The method of its solution using Dini series is outlined in Section 3. In Section 4, we present numerical results, which make clear the advantages of the present approach. Finally, in Section 5, some conclusions are derived.

2. Transformation of the integral equation

We start by transforming Eq. (2) in a more convenient form. Using the well-known property

$$\nabla' g(\mathbf{r},\mathbf{r}') = -\nabla g(\mathbf{r},\mathbf{r}') \tag{5}$$

the last integral in Eq. (2) is written as

$$\nabla \cdot \int_V \left[\frac{\rho_0}{\rho(\mathbf{r}')} - 1\right] \nabla'\Phi(\mathbf{r}') g(\mathbf{r},\mathbf{r}')\,\mathrm{d}V'. \tag{6}$$

The last integrand is further transformed using the identity

$$\mathbf{E} = \mathbf{I}\cdot\mathbf{E} = \mathbf{E}\cdot\mathbf{I}, \tag{7}$$

where $\mathbf{E}$ is a vector and $\mathbf{I}$ the identity dyadic. Finally, we get

$$\Phi(r) = \Phi^{\mathrm{inc}}(\mathbf{r}) + \int_V k_0^2 \left[\frac{b(\mathbf{r}')}{b_0} - 1\right] \Phi(\mathbf{r}') g(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' + \nabla \cdot \int_V \left[\frac{\rho_0}{\rho(\mathbf{r}')} - 1\right] \nabla'\Phi(\mathbf{r}') \cdot (\mathbf{I} g(\mathbf{r},\mathbf{r}'))\,\mathrm{d}V'. \tag{8}$$

The expansion of the Green's dyadic in spherical coordinates is given in Ref. [10, p. 1875] in terms of the even/odd spherical eigenvectors of the vector Helmholtz equation. Here, we use a more convenient form in terms of the complex form of these vectors as contained in Ref. [3]:

$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \mathbf{I}\frac{\mathrm{e}^{-\mathrm{i}k_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathrm{i}k_0}{4\pi}\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)}\sum_{m=-n}^{n}\frac{(n-m)!}{(n+m)!}\Big[\mathbf{M}_{mn}^{(1)}(k_0,r_<,\theta,\phi)\hat{\mathbf{M}}_{mn}^{(4)}(k_0,r_>,\theta',\phi')$$
$$+ \mathbf{N}_{mn}^{(1)}(k_0,r_<,\theta,\phi)\hat{\mathbf{N}}_{mn}^{(4)}(k_0,r_>,\theta',\phi') + n(n+1)\mathbf{L}_{mn}^{(1)}(k_0,r_<,\theta,\phi)\hat{\mathbf{L}}_{mn}^{(4)}(k_0,r_>,\theta',\phi')\Big]. \tag{9}$$

One can easily show that is equivalent to that of Ref. [10]. In Eq. (9), $r_>(r_<)$ denotes the larger (smaller) of r, r', while upperscripts (1), (4) imply the use of $j_n(k_0r)$, $h_n^{(2)}(k_0r)=j_n-\mathrm{i}n_n$, respectively, for the radial dependence of the eigenvectors. As in Ref. [3], the hat on the vectors implies the complex conjugate of the angular part only. The dyadic Green function in Eq. (9) is that of Morse and Feshbach [10], being in fact only the dyadic expansion of the scalar Green function $g(R)$. Thus, it does not contain any higher singularities than the last scalar function. So it should not be confused with the dyadic Green function used by Chew [3], defined as $(\mathbf{I}+\nabla\nabla)g(R)$, where the appearance of the del operators increases the order of singularity by 2, when both source and observation points are located at the same point. Using the surface harmonic function

$$Y_{mn}(\theta,\phi)=P_n^m(\cos\,\theta)\mathrm{e}^{\mathrm{i}m\phi}\quad (n=0,1,2,\ldots,\quad m=-n\text{ to }n), \tag{10}$$

we give below the definitions of the spherical eigenvectors denoting in general their radial spherical Bessel function by $z_n(kr)$:

$$\begin{aligned}\mathbf{L}_{mn}(k,r)&=\frac{1}{k}\nabla[z_n(kr)Y_{mn}(\theta,\phi)]\equiv\frac{1}{k}\nabla[\Psi_{mn}(k,r)]\\&=\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}r}[z_n(kr)]\mathbf{P}_{mn}(\theta,\phi)+\sqrt{n(n+1)}\frac{1}{kr}z_n(kr)\mathbf{B}_{mn}(\theta,\phi),\end{aligned} \tag{11}$$

$$\mathbf{M}_{mn}(k,\mathbf{r})=\frac{1}{k}\nabla\times[\mathbf{N}_{mn}(k,\mathbf{r})]=\sqrt{n(n+1)}z_n(kr)\mathbf{C}_{mn}(\theta,\phi), \tag{12}$$

$$\mathbf{N}_{mn}(k,\mathbf{r})=\frac{1}{k}\nabla\times[\mathbf{M}_{mn}(k,\mathbf{r})]=n(n+1)\frac{z_n(kr)}{kr}\mathbf{P}_{mn}(\theta,\phi)+\sqrt{n(n+1)}\frac{1}{kr}\frac{\mathrm{d}}{\mathrm{d}r}[rz_n(kr)]\mathbf{B}_{mn}(\theta,\phi). \tag{13}$$

In Eqs. (11)–(13) **P**, **B**, **C** are the surface spherical harmonic vectors which form an orthogonal and complete set over the spherical surface ($0\leqslant\theta\leqslant\pi,\quad 0\leqslant\phi\leqslant 2\pi$) and help separate the radial dependence of the **L**, **M**, **N** vectors from their angular one [10, pp.1898–1900]:

$$\mathbf{P}_{mn}(\theta,\phi)=\hat{\mathbf{r}}\,Y_{mn}(\theta,\phi), \tag{14}$$

$$\mathbf{B}_{mn}(\theta,\phi)=\hat{\mathbf{r}}\times\mathbf{C}_{mn}(\theta,\phi)=\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{n(n+1)}}\left[\frac{\mathrm{d}P_n^m(\cos\,\theta)}{\mathrm{d}\theta}\hat{\boldsymbol{\theta}}+\frac{\mathrm{i}m}{\sin\,\theta}P_n^m(\cos\,\theta)\hat{\boldsymbol{\phi}}\right], \tag{15}$$

$$\mathbf{C}_{mn}(\theta,\phi)=-\hat{\mathbf{r}}\times\mathbf{B}_{mn}(\theta,\phi)=\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{n(n+1)}}\left[\frac{\mathrm{i}m}{\sin\,\theta}P_n^m(\cos\,\theta)\hat{\boldsymbol{\theta}}-\frac{\mathrm{d}P_n^m(\cos\,\theta)}{\mathrm{d}\theta}\hat{\boldsymbol{\phi}}\right]. \tag{16}$$

From Eqs. (14) to (16), it is obvious that

$$\mathbf{P}_{mn}\cdot\mathbf{B}_{m'n'}=\mathbf{P}_{mn}\cdot\mathbf{C}_{m'n'}=\mathbf{B}_{mn}\cdot\mathbf{C}_{m'n'}=0, \tag{17}$$

while it is easily proved that

$$\int_0^{\pi}\int_0^{2\pi}\mathbf{X}_{mn}\cdot\hat{\mathbf{X}}_{m'n'}\,\sin\,\theta\,\mathrm{d}\phi\,\mathrm{d}\theta=\frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'}\delta_{mm'}, \tag{18}$$

where $\mathbf{X}=\mathbf{P}$ or **B** or **C**.

3. Solution of the integral equation

To solve the equation we expand the unknown field quantities in terms of scalar and vector wave functions in the interval $[0,a]$. Namely, expanding the radial part in Dini series, which constitute a full orthogonal set in $[0,a]$, we write [1,9]

$$\Phi(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{\ell=1}^{\infty}\left[A_{mn\ell}j_n\left(\frac{\gamma_{mn\ell}}{a}r\right)P_n^m(\cos\,\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right], \tag{19}$$

$$\left(\frac{b(\mathbf{r})}{b_0}-1\right)\Phi(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{\ell=1}^{\infty}\left[B_{mn\ell}j_n\left(\frac{\gamma_{mn\ell}}{a}r\right)P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right], \tag{20}$$

$$\left(\frac{\rho_0}{\rho(\mathbf{r})}-1\right)\nabla\Phi(\mathbf{r})=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{\ell=1}^{\infty}\left[\Gamma_{mn\ell}\mathbf{M}_{mn\ell}\left(\frac{\gamma^M_{mn\ell}}{a},\mathbf{r}\right)+\Delta_{mn\ell}\mathbf{N}_{mn\ell}\left(\frac{\gamma^N_{mn\ell}}{a},\mathbf{r}\right)+Z_{mn\ell}\mathbf{L}_{mn\ell}\left(\frac{\gamma^L_{mn\ell}}{a},\mathbf{r}\right)\right], \tag{21}$$

where $\gamma_{mn\ell}$ have been chosen as roots of the equation [1]

$$\gamma_{mn\ell}j_n'(\gamma_{mn\ell})/j_n(\gamma_{mn\ell})=-t_{mn}\quad(\ell=1,2,\ldots). \tag{22}$$

Here, t_{mn} is an arbitrary constant; in fact with different choices of t_{mn} we should obtain the same results. The sets $\mathbf{M}_{mn\ell}$ and $\mathbf{N}_{mn\ell}$ are included here for completeness. Their coefficients need not be calculated (see below). Anyway $\gamma^M_{mn\ell},\gamma^N_{mn\ell}$ and $\gamma^L_{mn\ell}$ have been chosen so as to construct a full orthogonal set of vectors **M**, **N**, and **L**, respectively, over the volume of the sphere $0\leqslant r\leqslant a$, $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi\leqslant 2\pi$. Moreover, all vectors $\mathbf{M}_{mn\ell},\mathbf{N}_{mn\ell}$, and $\mathbf{L}_{mn\ell}$ in those orthogonal relations are vectors of the first kind, i.e., $\mathbf{M}_{mn\ell}=\mathbf{M}^{(1)}_{mn\ell}$, with upperscript (1) deleted herein throughout. The details are found in Appendix A.

Furthermore, taking into account that the vectors $\mathbf{N}_{mn}$ and $\mathbf{L}_{mn}$ are not in general mutually orthogonal [11] over the surface of the sphere (i.e., in θ, ϕ), we have selected $\gamma^N_{mn\ell}$ as the roots of the indicial equation $j_n(\gamma^N_{mn\ell})=0$; this ensures the orthogonality between the $\mathbf{N}_{mn\ell}$ and $\mathbf{L}_{mn\ell}$ sets in the interval $0\leqslant r\leqslant a$. The details are given in Appendix A.

The calculation is carried out with the help of the following intermediate results:

$$I(\mathbf{M}\cdot\mathbf{G})=\int_V \mathrm{d}V'\mathbf{M}_{mn\ell}(k,\mathbf{r}')\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')$$
$$=\frac{1}{k^2-k_0^2}\{\mathbf{M}_{mn\ell}(k,\mathbf{r})-\mathrm{i}k_0a^2[-kj_n'(ka)h_n(k_0a)+k_0h_n'(k_0a)j_n(ka)]\mathbf{M}_{mn}(k_0,\mathbf{r})\}, \tag{23}$$

$$I(\mathbf{N}\cdot\mathbf{G})=\int_V \mathrm{d}V'N_{mn\ell}(k,\mathbf{r}')\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')=\frac{1}{k^2-k_0^2}\left\{\mathbf{N}_{mn\ell}(k,\mathbf{r})-\mathrm{i}k_0a^2\left[kj_n(ka)\frac{1}{k_0a}[xh_n(x)]'_{x=k_0a}\right.\right.$$
$$\left.\left.-k_0h_n(k_0a)\frac{1}{ka}\left[xj_n(x)\right]'_{x=ka}\right]\mathbf{N}_{mn}(k_0,\mathbf{r})\right\}-\mathrm{i}k_0a^2n(n+1)\frac{j_n(ka)h_n(k_0a)}{akk_0}\mathbf{L}_{mn}(k_0,\mathbf{r}), \tag{24}$$

$$I(\mathbf{L}\cdot\mathbf{G})=\int_V \mathrm{d}V'\mathbf{L}_{mn\ell}(k,\mathbf{r})\cdot\mathbf{G}(\mathbf{r},\mathbf{r}')=\frac{1}{k^2-k_0^2}\{\mathbf{L}_{mn\ell}(k,\mathbf{r})-\mathrm{i}k_0a^2[kj_n(ka)h_n'(k_0a)$$
$$-k_0h_n(k_0a)j_n'(ka)]\mathbf{L}_{mn}(k_0,\mathbf{r})\}-\mathrm{i}k_0a^2\frac{j_n(ka)h_n(k_0a)}{akk_0}\mathbf{N}_{mn}(k_0,\mathbf{r}). \tag{25}$$

In all the above equations $k=\gamma^X_{mnl}/a$, with $\mathbf{X}=\mathbf{M}$ or **N** or **L**, respectively. The derivation of these results is accomplished after much labor, using properties (14)–(18) of the surface spherical harmonic vectors **P**, **B**, **C**, and certain relations satisfied by the Bessel functions.

We next use the following differential equations satisfied by the **M**, **N**, **L** vectors:

$$\nabla\cdot\mathbf{M}=\nabla\cdot\mathbf{N}=0, \tag{26}$$

$$\nabla\cdot\mathbf{L}_{mn}=\frac{1}{k}\nabla^2\Psi_{mn}(k,\mathbf{r})=-k\Psi_{mn}(k,\mathbf{r}). \tag{27}$$

Here, it is obvious that the choice $j_n(\gamma^N_{mn\ell})=0$ is very convenient since, in addition to assuring orthogonality, it allows flexibility in the choice of the **L** set and leads to the disappearance of the **N** set. Therefore our previous assertion, that the **M** and **N** sets play no role in the solution, is confirmed and we can safely disregard them in what follows.

We now expand the incident field as in Ref. [1]

$$\Phi^{\mathrm{inc}}=\sum_{m,n}\left[A^{\mathrm{inc}}_{mn}j_n(k_0r)P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right]=\sum_{m,n,\ell}\left[A^{\mathrm{inc}}_{mn}T_{mn\ell}(k_0)j_n\left(\frac{\gamma_{mn\ell}}{a}r\right)P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}\right], \tag{28}$$

where $T_{mn\ell}(k_0)$ are obviously the expansions coefficients of $j_n(k_0 r)$ in the Dini series corresponding to the roots of Eq. (22)

$$j_n(k_0 r) = \sum_{\ell=1}^{\infty} T_{mn\ell}(k_0) j_n\left(\frac{\gamma_{mn\ell}}{a} r\right). \tag{29}$$

Substituting Eqs. (19)–(21) and (27) and (28), as well as the simple standard integral formulas for Bessel functions (Eqs. (14) and (15) of Ref. [1]) in Eq. (8) we end up with

$$\begin{aligned}
&\sum_{m,n,\ell}\left[A_{mn\ell} j_n\left(\frac{\gamma_{mn\ell}}{a} r\right) P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right] = \sum_{m,n,\ell}\left[A_{mn}^{\mathrm{inc}} T_{mn\ell}(k_0) j_n\left(\frac{\gamma_{mn\ell}}{a} r\right) P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}\right] \\
&+ k_0^2 \sum_{mn\ell} \frac{B_{mn\ell}}{(\gamma_{mn\ell}^M/a)^2 - k_0^2}\left\{ j_n\left(\frac{\gamma_{mn\ell}}{a} r\right) - \mathrm{i}k_0 a^2\left[-\frac{\gamma_{mn\ell}}{a} j_n'(\gamma_{mn\ell}) h_n(k_0 a) + k_0 h_n'(k_0 a) j_n(\gamma_{mn\ell})\right]\right. \\
&\left.\times \sum_p T_{mnp}(k_0) j_n\left(\frac{\gamma_{mnp}}{a} r\right)\right\} P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi} + \sum_{mn\ell} \frac{Z_{mn\ell}}{(\gamma_{mn\ell}^L/a)^2 - k_0^2}\left\{ -\frac{\gamma_{mn\ell}^L}{a} j_n\left(\frac{\gamma_{mn\ell}^L}{a} r\right) + \mathrm{i}k_0^2 a^2\right. \\
&\left.\times \left[\frac{\gamma_{mn\ell}^L}{a} j_n(\gamma_{mn\ell}^L) h_n'(k_0 a) - k_0 h_n(k_0 a) j_n'(\gamma_{mn\ell}^L)\right] \sum_p T_{mnp}(k_0) j_n\left(\frac{\gamma_{mnp}}{a} r\right)\right\} P_n^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}.
\end{aligned} \tag{30}$$

Finally, we obtain the system of equations

$$\begin{aligned}
A_{mn\ell} = {} & A_{mn}^{\mathrm{inc}} T_{mnl}(k_0) + k_0^2 \frac{B_{mn\ell}}{(\gamma_{mn\ell}^L/a)^2 - k_0^2} - \mathrm{i}k_0^3 a^2 T_{mn\ell}(k_0) \\
& \times \sum_p \frac{[-(\gamma_{mnp}/a) j_n'(\gamma_{mnp}) h_n(k_0 a) + k_0 h_n'(k_0 a) j_n(\gamma_{mnp})]}{(\gamma_{mnp}/a)^2 - k_0^2} B_{mnp} - \frac{(\gamma_{mn\ell}/a) Z_{mn\ell}}{(\gamma_{mn\ell}/a)^2 - k_0^2} \\
& + \mathrm{i}k_0^2 a^2 T_{mn\ell}(k_0) \sum_p \frac{[(\gamma_{mnp}/a) j_n(\gamma_{mnp}) h_n'(k_0 a) - k_0 h_n(k_0 a) j_n'(\gamma_{mnp})]}{(\gamma_{mnp}/a)^2 - k_0^2} Z_{mnp}.
\end{aligned} \tag{31}$$

We now correlate the group of unknowns. The relationship between $\{A\}$ and $\{B\}$ is obvious and simple as in Ref. [1]:

$$B_{mn\ell} = \frac{1}{N_{mn\ell}} \sum_{\mu\nu\lambda} Q(\Psi_{\mu\nu\lambda}, \Psi_{mn\ell}^*) A_{\mu\nu\lambda} \tag{32}$$

in which

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^a \left(\frac{b(\mathbf{r})}{b_0} - 1\right) j_n\left(\frac{\gamma_{mn\ell}}{a} r\right) j_\nu\left(\frac{\gamma_{\mu\nu\lambda}}{a} r\right) P_n^m(\cos\theta) P_\nu^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}(\mu-m)\phi} r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{33}$$

and

$$N_{mn\ell} = \frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\int_0^a j_n^2\left(\frac{\gamma_{mn\ell}}{a} r\right) r^2\,\mathrm{d}r = \frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!} a^3 j_n^2(\gamma_{mn\ell})\left[1 - \frac{(n+1-t_n)(n+t_n)}{\gamma_{mn\ell}^2}\right], \tag{34}$$

while for the relationship between $\{Z\}$ and $\{A\}$ set we write formally

$$\left(\frac{\rho_0}{\rho(\mathbf{r})} - 1\right)\nabla\Phi = \left(\frac{\rho_0}{\rho(\mathbf{r})} - 1\right)\sum_{mn\ell}\left[\frac{\gamma_{mn\ell}}{a} A_{mn\ell} \mathbf{L}_{mn\ell}^S\left(\frac{\gamma_{mn\ell}}{a}, \mathbf{r}\right)\right]. \tag{35}$$

Here, serious questions arise. First, the differentiation of the Dini series is in general not allowed [1,9]. In our calculations, however, the derivative of a function $f(r)$ that is being expanded appears inside an integral. Let us consider a typical one:

$$T = \int r'^2\,\mathrm{d}r' \frac{\mathrm{d}f}{\mathrm{d}r'}\left(\frac{\rho_0}{\rho(r')} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>), \tag{36}$$

which after integration by parts can be written

$$T = r^2 f(r)\left(\frac{\rho_0}{\rho(r)} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>) - \int f(r) \frac{\mathrm{d}}{\mathrm{d}r'}\left[r'^2\left(\frac{\rho_0}{\rho(r')} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>)\right] \mathrm{d}r'. \tag{37}$$

Here, we can substitute $f(r)$ by a Dini series

$$f(r) = \sum_{\mu=1}^{\infty} \alpha_\mu j_n\left(\frac{\gamma_{n\mu}}{a} r\right) \tag{38}$$

and since the series converges as $1/\mu^2$ we may change the order of summation and integration to obtain

$$\begin{aligned} T = &\sum_\mu \alpha_\mu r^2 j_n\left(\frac{\gamma_{n\mu}}{a} r\right)\left(\frac{\rho_0}{\rho(r')} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>) \\ &- \sum_\mu \alpha_\mu \int j_n\left(\frac{\gamma_{n\mu}}{a} r\right) \frac{\mathrm{d}}{\mathrm{d}r'}\left[r'^2\left(\frac{\rho_0}{\rho(r')} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>)\right] \mathrm{d}r'. \end{aligned} \tag{39}$$

Next reversing the integration by parts we get

$$T = \sum_\mu \alpha_\mu \int r'^2 \frac{\mathrm{d}j_n((\gamma_{n\mu}/a)r)}{\mathrm{d}r'}\left(\frac{\rho_0}{\rho(r')} - 1\right) j_n(k_0 r_<) h_n(k_0 r_>)\,\mathrm{d}r'. \tag{40}$$

The last series converges with μ at least as $1/\mu^3$ [1]. So although the differentiation of the Dini series is not allowed it can safely be used for the calculation of the above integral, in the manner indicated: that is first integrating and then summing the resulting terms as shown in Eq. (40). On the other hand, it would have been erroneous to calculate $\mathrm{d}f/\mathrm{d}r$ first from the Dini series and then use the result to carry out the integration in Eq. (36).

Secondly, as seen from Eqs. (22) and (A.10), the $\mathbf{L}^S$ vectors that result from the differentiation of Ψ are not, in general, orthogonal themselves. One can try to work with the non-orthogonal $\mathbf{L}^S$ set, but this leads to difficulties. So, it is necessary to use an orthogonal $\mathbf{L}$ set (and especially the one that has already been incorporated in Eq. (21)), different from the non-orthogonal $\mathbf{L}^S$ set (where $\mathbf{L}^S_{mn\ell}((\gamma_{mn\ell}/a)r) = (a/\gamma_{mn\ell})\nabla\Psi_{mn\ell}$).

They are related as follows:

$$\mathbf{L}^S_{mn\ell} = \sum_{\mu,\nu,\lambda}[S(\mathbf{L}^S_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu\lambda})\mathbf{L}_{\mu\nu\lambda}] \tag{41}$$

with

$$\mathbf{S}(\mathbf{L}^S_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu\lambda}) = \frac{I(\mathbf{L}^S_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu\lambda})}{I(\mathbf{L}_{\mu\nu\lambda}, \hat{\mathbf{L}}_{\mu\nu\lambda})}. \tag{42}$$

The I's are defined in Appendix A. Next, we expand the terms $((\rho_0/\rho(\mathbf{r})) - 1)\mathbf{L}_{mn\ell}$ in the orthogonal $\mathbf{L}_{mn\ell}$ set

$$\left[\left(\frac{\rho_0}{\rho(\mathbf{r})} - 1\right)\mathbf{L}_{mn\ell}\right]_L = \sum_{\mu\nu\lambda} \frac{W(\mathbf{L}_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu\lambda})}{I(\mathbf{L}_{\mu\nu\lambda}, \hat{\mathbf{L}}_{\mu\nu\lambda})}\mathbf{L}_{\mu\nu\lambda} \equiv \sum_{\mu\nu\lambda} F_{\mu\nu\lambda}\mathbf{L}_{\mu\nu\lambda} \tag{43}$$

with

$$W(\mathbf{X}, \hat{\mathbf{Y}}) = \int_V \mathrm{d}V'\left(\frac{\rho_0}{\rho(\mathbf{r})} - 1\right)\mathbf{L}_{mn\ell}\left(\frac{\gamma^L_{mn\ell}}{a}, \mathbf{r}\right) \cdot \hat{\mathbf{L}}_{\mu\nu\lambda}\left(\frac{\gamma^L_{\mu\nu\lambda}}{a}, \mathbf{r}\right). \tag{44}$$

In general, when ρ depends on θ, ϕ, expansion (43) should contain also $\mathbf{M}$ and $\mathbf{N}$ vectors. However, as before, due to the del operator outside the integral any such term will disappear. Next after the integration, the application of the $\nabla\cdot$ operator gives rise to a scalar set, which is non-orthogonal and should be re-expanded in terms of the original scalar orthogonal Dini set. This step, however, can be avoided by simply re-expanding the orthogonal $\mathbf{L}_{mn\ell}$ set in the non-orthogonal $\mathbf{L}^S_{mn\ell}$ using the inverse $\mathbf{S}^{-1}$ of the matrix in Eq. (42). So, finally, we get the following matrix relation:

$$\mathbf{Z} = \mathbf{S}^{-1}\mathbf{F}\mathbf{S}(\mathbf{A}\mathbf{x}), \tag{45}$$

where $\mathbf{Z}$, $\mathbf{Ax}$ are column vectors with elements $Z_{mn\ell}$, $A_{mn\ell}(\gamma_{mn\ell}/a)$, respectively, while the elements of $\mathbf{F}$ have been defined in Eq. (43).

Apparently, a simpler choice would have been to choose $j_n(\gamma_{mn\ell}) = j_n(\gamma^L_{mn\ell}) = 0$ as common roots for both sets. However, contrary to the general $1/\ell^2$ rate of convergence of Dini series [1], further calculations with the above choice show that the convergence deteriorates to $1/\ell^{1+\varepsilon}$ with $\varepsilon > 0$, a small number, rendering the whole solution extremely unstable and very sensitive to truncation size.

Finally, we obtain a system of equations for the unknown expansion coefficients $A_{mn\ell}$, which is solved by truncation.

4. Numerical results and discussion

The solution we derive in this manner was compared with the one obtained with the method previously developed in Ref. [2]. In the cases we tested $\rho(r) = \rho_1 = \text{const}$, $\rho(r) = 20\rho_0 \exp(-200r^2)$ and $\rho(r) = \rho_0(3 + 2\cos(\pi r/a)$ a very good agreement between the two methods has been obtained. Especially for the case $\rho(r) = \rho_1 = \text{constant}$ a simple analytical treatment is possible, producing the correct result and thus eliminating any doubt on the validity of Eqs. (2) and (8). The procedure is given in Appendix B. These cases, as well as others in which the density does not contain step discontinuities dependent on θ are treated more easily with the use of the delta function approach in Ref. [2]. Actually, if such was the case, it could have been solved through the use of another Lippmann–Schwinger integral equation [12]. Namely, the incident field could have been chosen as the solution of the problem for a sphere with constant density equal to the density just inside the inhomogeneity. Thus, no discontinuity would have appeared and no use of for delta functions would be necessary.

However, the strength of the present approach is obvious when a more complicated case is considered. Let us consider, for instance, the case

$$\rho(r,\theta) = \frac{\rho_0}{(2/3) + (1/3)(r/a)^2 \cos\theta}, \quad b(r,\theta) = 2b_0 \tag{46}$$

that is a case where the inhomogeneity across the boundary of the sphere depends on θ. This problem has been treated in Ref. [2] in a rather complicated manner, by using as additional unknowns the values of the expansions coefficients just inside the inhomogeneity. This approach, although formally correct, complicates the whole procedure, and it becomes prohibitively complex when $\rho(\mathbf{r})$ contains internal discontinuities dependent on θ.

In Figs. 2–4, the total internal acoustic pressure is plotted for an incident plane wave from the z direction, with $k_0a = 12.5751$, 16.7668 and 20.9584, respectively. We have chosen the $\rho(\mathbf{r})$ given by Eq. (46) while $b(\mathbf{r}) = 2b_0$ for simplicity. Here, we have worked without any optimum t_{mn} and we had no results from the literature to compare with. However, we have obtained exactly the same field results using other random choices for t_{mn}; we consider this as a confirmation of our approach in this general case.

To demonstrate the efficiency of the method we have included results in which the values of n and ℓ in the infinite series (19)–(21) have been truncated to finite values $n_{\max}$ and $\ell_{\max}$, respectively. So we present in Table 1 the truncation parameters for the case described in Eq. (46) for various values of k_0a, all for 1% accuracy, that is we find the values at which the maximum relative difference for the field values Φ from those obtained from higher values of n and ℓ are less than 1%. For the simpler r-dependent case the final values so obtained are not essentially different from the ones presented in Refs. [1,2]. However, when $\rho(r)$ is discontinuous at the boundary $r = a$ the values of $\ell_{\max}$ should be taken a little greater.

As far as the applicability of the method is concerned, the analytic evaluation of the coefficients of the algebraic matrix system is possible here due to the spherical coordinate system. It can also be done in the circular cylindrical one, in the case of two-dimensional scattering. In other coordinate systems, the more complex form of the Green function expansion (if it exists at all), as well as the absence of suitable orthogonal sets, like that of Dini series, may hinder or even make impossible the applicability of the present approach.

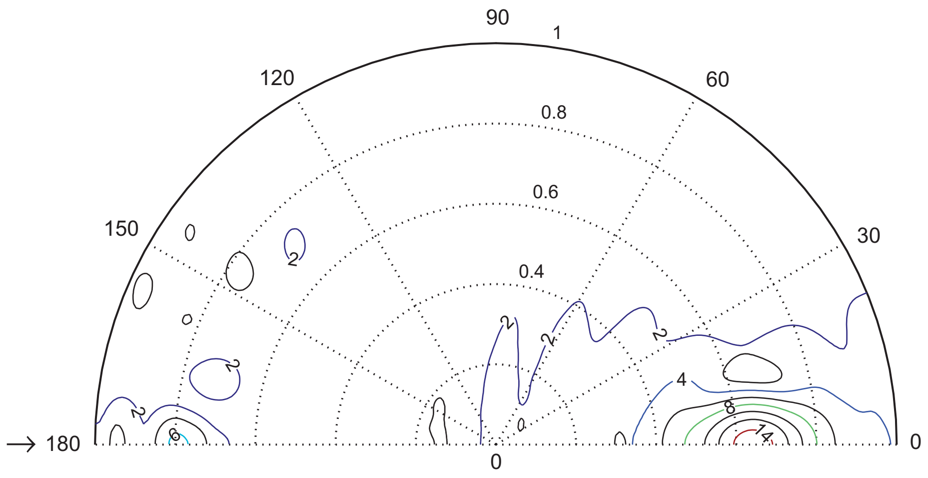

Fig. 2. $|\Phi|$ for $\rho(r,\theta) = \rho_0/(2/3) + (1/3)(r/a)^2 \cos\theta$, $b(r) = 2b_0$, $k_0a = 12.5751$ for various θ, incident field plane wave from z direction.

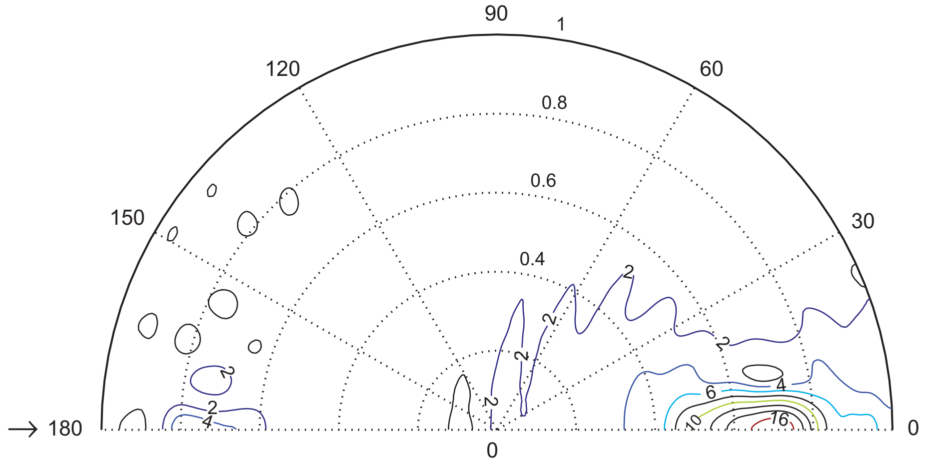

Fig. 3. $|\Phi|$ for $\rho(r,\theta) = \rho_0/(2/3) + (1/3)(r/a)^2 \cos\theta$, $b(r) = 2b_0$, $k_0a = 16.7668$ for various θ, incident field plane wave from z direction.

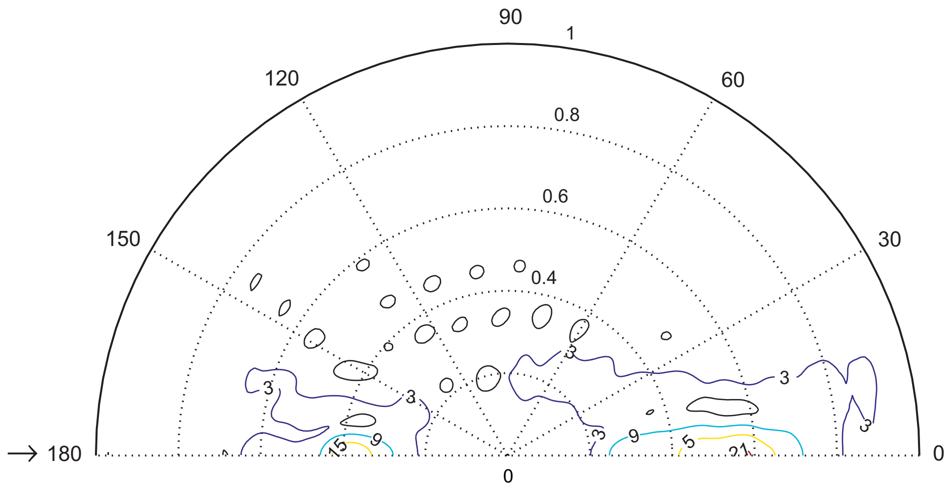

Fig. 4. $|\Phi|$ for $\rho(r,\theta) = \rho_0/(2/3) + (1/3)(r/a)^2 \cos\theta$, $b(r) = 2b_0$, $k_0a = 20.9584$ for various θ, incident field plane wave from z direction.

Table 1
Number of truncation parameters for 1% accuracy in the case of Eq. (46)

k_0a	$n_{\max}$	$\ell_{\max}$
2.0958	4	8
4.1917	10	14
8.3834	16	19
12.5751	21	23
16.7668	25	27
20.9584	28	30

A peculiar advantage that makes the whole procedure feasible here is the existence of indefinite integrals [1,13] of Bessel functions of the same order but different argument. This seems to limit the procedure to the previous systems only. However, the whole procedure may be applied in other shapes (non-circular cylindrical or non-spherical) with the aid of the recently developed conformal mapping technique [14,15]. As it is explained there, the PDE for a non-circular/spherical scatterer is transformed with the aid of a suitably chosen mapping function to a PDE for a non-homogeneous circular/spherical scatterer. However, this point is a matter of further research.

Finally, the method can be applied to other type of inhomogeneous (especially graded) materials of spherical shape like elastic, piezoelectric [16], or piezoceramic [17]. The present method, based on integral equations can be considered as an alternative to differential equation methods like that of Frobenius [18].

5. Conclusion

Acoustic fields induced by external sources in media with inhomogeneous density and compressibility have been treated by modifying the direct hybrid (analytical–numerical) method developed previously [1,2]. The modification provides a unified treatment of all cases, regardless of possible step discontinuities in the density. This was accomplished by using a different volume integral equation that has been derived by Ivakin, extracting once out of the volume integral the one differential operator and by using the Green's dyadic in connection with an expansion of the derivative of the pressure field in terms of **L** vectors. The final algebraic equation is quite simple. The matrix sizes required are not larger than those of the previous approach; so the present approach is a powerful method that can tackle any acoustic field with radial dependence described by a Dini series; even the case of internal step discontinuities is also easily treated without resorting to the use of delta functions with their inherent analytical complications that were apparent in the previous approach [2].

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Appendix A. Selection of the proper orthogonal sets $\mathbf{M}_{mn\ell}$, $\mathbf{N}_{mn\ell}$, $\mathbf{L}_{mn\ell}$

The construction of the proper sets of vectors $\mathbf{M}_{mn\ell}$, etc., fully orthogonal over the volume of the sphere, as mentioned in connection with Eq. (21), is based on the orthogonal properties of the surface spherical harmonics vectors $\mathbf{P}_{mn}$, $\mathbf{B}_{mn}$, $\mathbf{C}_{mn}$ and a suitable choice of the arguments of the Bessel functions, expressing the radial dependence of the vectors, in a way similar to that used for the corresponding Dini's expansions in the scalar case [1,2]. For the **M** set, the procedure leads, via Eq. (10), directly to the integral

$$I(\mathbf{M}_{mn\ell},\hat{\mathbf{M}}_{\mu\nu p}) = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \mathbf{M}_{mn\ell}\left(\frac{\gamma^M_{mn\ell}}{a},\mathbf{r}\right)\cdot\hat{\mathbf{M}}_{\mu\nu p}\left(\frac{\gamma^M_{\mu\nu p}}{a},\mathbf{r}\right) r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$= 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{m\mu}\delta_{n\nu}\int_0^a j_n\left(\frac{\gamma^M_{mn\ell}}{a}r\right)j_n\left(\frac{\gamma^M_{mnp}}{a}r\right)r^2\,\mathrm{d}r$$

$$= 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{m\mu}\delta_{n\nu}\frac{a^3 j_n(\gamma^M_{mn\ell})j_n(\gamma^M_{mnp})}{(\gamma^M_{mn\ell})^2-(\gamma^M_{mnp})^2}\left[\frac{\gamma^M_{mn\ell}j'_n(\gamma^M_{mn\ell})}{j_n(\gamma^M_{mn\ell})}-\frac{\gamma^M_{mnp}j'_n(\gamma^M_{mnp})}{j_n(\gamma^M_{mnp})}\right],\quad \ell\neq p. \tag{A.1}$$

Similar relations were found in Refs. [1,2]. We can now establish full orthogonality of the set over the volume of the sphere by selecting $\gamma^M_{mn\ell}$ as the roots of the "*M*-eigenvalue equation"

$$\frac{\gamma^M_{mn\ell}j'_n(\gamma^M_{mn\ell})}{j_n(\gamma^M_{mn\ell})}\equiv t^M_{nm}\quad (\ell=1,2,\ldots) \tag{A.2}$$

in which t^M_{nm} may be any chosen constant. Moreover, the index m in Eq. (A.2) is not deleted, in order to imply that for each m a different choice of t_{nm} and $\gamma_{mn\ell}$ is possible. As before [1,2] a proper choice of t^M_{nm} helps improve the convergence of series expansions over $\ell=1,2,\ldots$. With such values for $\gamma^M_{mn\ell}$ (A.1) becomes (see also Ref. [1])

$$I(\mathbf{M}_{mn\ell},\hat{\mathbf{M}}_{\mu\nu p}) = 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\frac{\alpha^3}{2}[j_n^2(\gamma^M_{mn\ell})-j_{n-1}(\gamma^M_{mn\ell})j_{n+1}(\gamma^M_{mn\ell})]\delta_{m\mu}\delta_{n\nu}\delta_{\ell p}. \tag{A.3}$$

The orthogonality of the above **M** set to any **N** or **L** vectors from the other sets follows directly from their θ, ϕ dependence, as in Stratton [11].

Following the same procedure for the **N** set, using Eq. (13) and the abbreviations $\gamma^M_{mn\ell}/a=k^N_\ell$, $\gamma^M_{mnp}/a=k^N_p$ we get

$$I(\mathbf{N}_{mn\ell},\hat{\mathbf{N}}_{\mu\nu p}) = \int_V \mathrm{d}V'\mathbf{N}_{mn\ell}(k^N_\ell,\mathbf{r})\cdot\hat{\mathbf{N}}_{\mu\nu p}(k^N_p,\mathbf{r}) = 4\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{m\mu}\delta_{n\nu}\int_0^a T_N r^2\,\mathrm{d}r, \tag{A.4}$$

where

$$\int_0^a T_N r^2\,\mathrm{d}r = \frac{n(n+1)}{k^N_\ell k^N_p}\int_0^a j_n(k^N_\ell r)j_n(k^N_p r)\,\mathrm{d}r + \frac{1}{k^N_\ell k^N_p}\int_0^a \frac{\mathrm{d}}{\mathrm{d}r}[rj_n(k^N_\ell r)]\frac{\mathrm{d}}{\mathrm{d}r}[rj_n(k^N_p r)]\,\mathrm{d}r$$

$$= \frac{n(n+1)}{k^N_\ell k^N_p}\int_0^a j_n(k^N_\ell r)j_n(k^N_p r)\,\mathrm{d}r + \frac{1}{k^N_\ell k^N_p}\left\{aj_n(k^N_\ell a)[j_n(k^N_p a)+k^N_p a j'_n(k^N_p a)]\right.$$

$$\left. -\int_0^a rj_n(k^N_\ell r)[2k^N_p j'_n(k^N_p r)+(k^N_p)^2 r j''_n(k^N_p r)]\right\}\mathrm{d}r. \tag{A.5}$$

Using the Bessel differential equation, we substitute

$$j''_n(x) = -\frac{2}{x}j'_n(x)+\left[\frac{n(n+1)}{x^2}-1\right]j_n(x)$$

to obtain

$$\int_0^a T_N r^2\,\mathrm{d}r = \frac{1}{k^N_\ell k^N_p}\left\{aj_n(k^N_\ell a)[xj_n(x)]'_{x=\gamma^N_{mn\ell}}+(k^N_p)^2\int_0^a r^2 j_n(k^N_\ell r)j_n(k^N_p r)\,\mathrm{d}r\right\}$$

and the end result is

$$\int_0^a T_N r^2\,\mathrm{d}r = \frac{ak^N_\ell k^N_p j_n(k^N_\ell a)j_n(k^N_p a)}{(k^N_p)^2-(k^N_\ell)^2}\left\{\frac{[k^N_p a j_n(k^N_p a)]'}{(k^N_p)^2 j_n(k^N_p a)}-\frac{[k^N_\ell a j_n(k^N_\ell a)]'}{(k^N_\ell)^2 j_n(k^N_\ell a)}\right\}\quad(\ell\neq p). \tag{A.6}$$

Orthogonality of the **N**-set over the volume of the sphere is then established if we choose $\gamma^N_{mn\ell}$ as the roots of the "*N*-eigenvalue" equation"

$$\frac{[\gamma^N_{mn\ell} j_n(\gamma^N_{mn\ell})]'}{(\gamma^N_{mn\ell})^2 j_n(\gamma^N_{mn\ell})} \equiv t^N_{nm} \quad (\ell = 1, 2, \ldots). \tag{A.7}$$

Then,

$$I(\mathbf{N}_{mn\ell}, \hat{\mathbf{N}}_{\mu\nu p}) = 4\pi \frac{n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} \frac{\alpha^3}{2} \left\{ j_n^2(\gamma^N_{mn\ell}) \left[1 - \frac{n(n+1)-2}{(\gamma^N_{mn\ell})^2} \right] + j_n'^2(\gamma^N_{mn\ell}) + 3 j_n'(\gamma^N_{mn\ell}) \frac{j_n(\gamma^N_{mn\ell})}{\gamma^N_{mn\ell}} \right\} \delta_{m\mu} \delta_{n\nu} \delta_{\ell p} \tag{A.8}$$

Following the same procedure with the **L**-set of vectors we get

$$I(\mathbf{L}_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu p}) = \int_V \mathrm{d}V' \mathbf{L}_{mn\ell}(k^L_\ell, \mathbf{r}) \cdot \hat{\mathbf{L}}_{\mu\nu p}(k^L_p, \mathbf{r}) = 4\pi \frac{1}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{m\mu} \delta_{n\nu}$$
$$\times \frac{a^2 k^L_\ell k^L_p j_n(k^L_\ell a) j_n(k^L_p a)}{(k^L_\ell)^2 - (k^L_p)^2} \left[\frac{j_n'(k^L_p a)}{k^L_p j_n(k^L_p a)} - \frac{j_n'(k^L_\ell a)}{k^L_\ell j_n(k^L_\ell a)} \right] \quad (\ell \neq p). \tag{A.9}$$

Orthogonality of the **L**-set over the volume of the sphere is then established if $\gamma^L_{mn\ell}$ are chosen as the roots of the "*L*-eigenvalue equation"

$$\frac{j_n'(\gamma^L_{mn\ell})}{\gamma^L_{mn\ell} j_n(\gamma^L_{mn\ell})} \equiv t^L_{nm} \quad (\ell = 1, 2, \ldots). \tag{A.10}$$

Then,

$$I(\mathbf{L}_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu p}) = 4\pi \frac{1}{2n+1} \frac{(n+m)!}{(n-m)!} \frac{\alpha^3}{2} \left\{ j_n^2(\gamma^L_{mn\ell}) \left[1 - \frac{n(n+1)}{(\gamma^L_{mn\ell})^2} \right] + j_n'^2(\gamma^L_{mn\ell}) + 3 j_n'(\gamma^L_{mn\ell}) \frac{j_n(\gamma^L_{mn\ell})}{\gamma^L_{mn\ell}} \right\} \delta_{m\mu} \delta_{n\nu} \delta_{\ell p} \tag{A.11}$$

Finally, we must consider the orthogonality between the **L** and **N** sets, which is not assured from their angular (θ, ϕ) part only. However, over the volume of the sphere we have

$$I(\mathbf{N}_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu p}) = \int_V \mathrm{d}V' \mathbf{N}_{mn\ell}(k^N_\ell, \mathbf{r}) \cdot \hat{\mathbf{L}}_{\mu\nu p}(k^L_p, \mathbf{r}) = 4\pi \frac{n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{m\mu} \delta_{n\nu} \int_0^a T_{NL} r^2 \, \mathrm{d}r, \tag{A.12}$$

where from Eqs. (11) and (13), and integration by parts we get for all values of ℓ, p

$$\int_0^a T_{NL} r^2 \, \mathrm{d}r = \int_0^a \left\{ j_n'(k^L_p r) \frac{j_n(k^N_\ell r)}{k^N_\ell r} + \frac{1}{k^L_p r} j_n(k^L_p r) \frac{1}{k^N_\ell r} \frac{\mathrm{d}}{\mathrm{d}r} [r j_n(k^N_\ell r)] \right\} r^2 \, \mathrm{d}r = \frac{a^3}{\gamma^N_{mn\ell} \gamma^L_{mnp}} j_n(\gamma^N_{mn\ell}) j_n(\gamma^L_{mnp}). \tag{A.13}$$

Orthogonality and decoupling between the **L** and **N** sets is now established over the volume of the sphere if we choose as $\gamma^N_{mn\ell}$ or γ^L_{mnp} the roots of either $j_n(\gamma^N_{mn\ell}) = 0$ or $j_n(\gamma^L_{mnp}) = 0$. The first choice is more convenient here, since it leaves $\gamma^L_{mn\ell}$ unchanged from our previous choice (A.10). So now, $I(\mathbf{N}_{mn\ell}, \hat{\mathbf{L}}_{\mu\nu\mathrm{p}}) = 0$ even when $\ell = p$.

Appendix B

In the case of a sphere of constant density $\rho_1 \neq \rho_0$ (as well as of constant compressibility $b_1 \neq b_0$), integral equation (4) takes the form

$$\Phi(\mathbf{r}) = \Phi^{\mathrm{inc}}(\mathbf{r}) + k_0^2 \left[\frac{b_1}{b_0} - 1 \right] \int_V \Phi(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') \, \mathrm{d}V' - \left[\frac{\rho_0}{\rho_1} - 1 \right] \int_V \nabla' \Phi(\mathbf{r}') \cdot \nabla' g(\mathbf{r}, \mathbf{r}') \, \mathrm{d}V'. \tag{B.1}$$

Let us try a solution of the form

$$\Phi(\mathbf{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} [A_{mn} j_n(kr) P_n^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi}]. \tag{B.2}$$

Here, there is no need for extra expansion like that described in Eq. (43). We work with $\mathbf{L}_{mn}(k,\mathbf{r}) = (1/k)\nabla[j_n(kr)Y_{mn}(\theta,\phi)]$. In fact using Eqs. (25), (27), the first of Eq. (28), as well as Eqs. (14) and (15) of Ref. [1] we end up with

$$A_{mn}j_n(kr) = A_{mn}^{\text{inc}}j_n(k_0r) + \rho_0\frac{\omega^2(b_1-b_0)}{k^2-k_0^2}A_{mn}\{j_n(kr) + \mathrm{i}k_0a^2j_n(k_0r)[kj_n'(ka)h_n(k_0a) - k_0j_n(ka)h_n'(k_0a)]\}$$
$$+ A_{mn}\left[\frac{\rho_0}{\rho_1}-1\right]\frac{k}{k^2-k_0^2}\{-kj_n(kr) + \mathrm{i}k_0^2a^2j_n(k_0r)[kj_n(ka)h_n'(k_0a) - k_0h_n(k_0a)j_n'(ka)]\}. \tag{B.3}$$

If we choose $k = k_1 = \omega\sqrt{b_1\rho_1}$ the terms of $j_n(k_1r)$ are cancelled out and we obtain

$$A_{mn} = -\frac{1/\mathrm{i}k_0a^2}{(\rho_0/\rho_1)kj_n'(ka)h_n(k_0a) - k_0j_n(ka)h_n'(k_0a)} \tag{B.4}$$

the same expression as that found with the separation of variables method.

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